Translational dynamics effects on the non-local correlations between two atoms

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A pair of atoms interacting successively with the field of the same cavity and exchanging a single photon, leave the cavity in an entangled state of Einstein-Podolsky-Rosen (EPR) type (see, for example, [S.J.D. Phoenix, and S.M. Barnett, J. Mod. Opt. **40** (1993) 979]). By implementing the model with the translational degrees of freedom, we show in this letter that the entanglement with the translational atomic variables can lead, under appropriate conditions, towards the separability of the internal variables of the two atoms. This implies that the translational dynamics can lead, in some cases, to difficulties in observing the Bell's inequality violation for massive particles.

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A necessary condition for attaining fruitful teleportation [1] is the possibility of constructing "non locally" correlated systems. Recently, it has been payed attention to teleportation of massive particles and, more generally, to non-local correlations, separability and related issues [2, 3, 4, 5, 6, 7, 8]. A simple model which can realize an EPR state consists of two atoms which interact successively with the field of an optical cavity. As a consequence of the entanglement developed during the interaction between the first atom and the field, quantum correlations between the two atoms do arise as the second one interacts with the field of the same cavity. Using the standard Jaynes-Cummings (JC) model, non-local correlations have been predicted [4] which can lead to a violation of the Bell's inequality.
In this letter we suggest that a careful analysis of the interatomic correlations may require the quantization of the translational dynamics of the two atoms along the cavity axis. In fact, it is here shown that the quantization of the translational dynamics affects the non-local features of the interatomic correlations, at least when the atoms enter the cavity in a region where the field gradient is different from zero (for example in a nodal region). To take into account these translational effects, we adopt the optical Stern-Gerlach (SG) model [9] for the atom-field interaction, while to investigate on the non-local features we test the Bell's inequality [10] and the separability [11, 12] for the reduced density matrix describing the internal degrees of freedom of the two atoms, after their interaction with the cavity. For this system our main quantitative result reveals the appearance of damping terms in the correlation functions of the two-atom internal variables, induced by the entanglement with the translational dynamics. These damping factors destroy the quantum nature of the correlations just for an interaction time of a few Rabi oscillations.
Let us consider a system composed by two two-level atoms interacting, not simultaneously, with the e.m. field of the same undamped cavity. At the time $t = 0$, the first atom, say A_1, enters the cavity moving prevalently along the z-direction, orthogonal to the x-cavity axis, and interacts with the field for a time t_1. We assume that the atomic velocity along the z-direction is large enough to treat classically this component of the motion. At a later time t_2, the second atom, say A_2, starts interacting with the field-state in the cavity modified by the first atom. At the time t_3, A_2 leaves the cavity and both the atoms evolve freely. We assume that the atoms enter the cavity in proximity of a nodal region of the cavity resonant mode k, and the width of their wave packets is sufficiently small with respect to the wavelength of the k-mode. In the interaction picture, the Hamiltonian of the system at all times reads

$$\hat{H}^I(t) = \hbar\varepsilon k(\hat{x}_1 + \frac{\hat{p}_1}{m}t)\mu_t(0,t_1)\hat{u}_1 + \hbar\varepsilon k(\hat{x}_2 + \frac{\hat{p}_2}{m}t)\mu_t(t_2,t_3)\hat{u}_2, \tag{1}$$

where the observable $\hat{x}_i$ describes the position of A_i with respect to the nodal point and $\hat{p}_i$ is the conjugate momentum. The atom-field interaction operator $\hat{u}_i = \hat{a}^\dagger\hat{S}_{-,i} + \hat{a}\hat{S}_{+,i}$ mixes the annihilation and creation field-operators $\hat{a}$ and $\hat{a}^\dagger$ with the usual spin 1/2 operators $\hat{S}_{\pm,i}$. Finally $\mu_t(x,y) = \theta_t(x) - \theta_t(y)$ is a linear combination of the usual step-functions with different points of discontinuity (x and y), helpful to distinguish the different time-ranges concerning the successive interaction of the two atoms (with same mass m and same atom-field coupling constant ε) with the cavity-field.
To evaluate the state of the entire system at some time $t \geq t_3$, when both the atoms are outside the cavity, we look at the evolution operator (for the one atom case see Ref.[13])

$$\begin{aligned}\hat{U}^I(t \geq t_3) = &\exp\{-i\varepsilon k\hat{u}_2(t_3 - t_2)[\hat{x}_2 + \frac{\hat{p}_2}{2m}(t_3 + t_2)]\} \cdot \\ &\cdot \exp[i\hbar\frac{\varepsilon^2k^2}{12m}\hat{u}_2^2(t_3 - t_2)^3] \exp[i\hbar\frac{\varepsilon^2k^2}{12m}\hat{u}_1^2t_1^3] \exp[-i\varepsilon k\hat{u}_1t_1(\hat{x}_1 + \frac{\hat{p}_1}{2m}t_1)].\end{aligned} \tag{2}$$

For ease of comparison with the results obtained in the ambit of the standard JC model [4], we consider the simple case in which, initially, both the two atoms are in the ground state $|g_i\rangle$, and the cavity mode contains just one photon,

$$|\psi(0)\rangle = |g_1\rangle\,|g_2\rangle\,|1\rangle\,|\varphi_1(0)\rangle\,|\varphi_2(0)\rangle\,, \tag{3}$$

where $|\varphi_i(0)\rangle$ is a translational state of the atom A_i. At a time $t \geq t_3$, after the successive atom-field interactions, the entire system state is

$$|\psi(t_3)\rangle = [\left|S_1^-\right\rangle|\varphi_2(0)\rangle\,|e_1\rangle\,|g_2\rangle + \left|S_1^+\right\rangle\left|S_2^-\right\rangle|g_1\rangle\,|e_2\rangle]\,|0\rangle + \left|S_1^+\right\rangle\left|S_2^+\right\rangle|g_1\rangle\,|g_2\rangle\,|1\rangle\,, \tag{4}$$

where $|e_i\rangle$ indicates the excited state of A_i. The atomic translational parts modified because of the interaction with the cavity-photons are

$$\left|S_i^{\pm}\right\rangle = \frac{1}{2}[\left|\phi_i^+\right\rangle \pm \left|\phi_i^-\right\rangle], \tag{5}$$

where $i\epsilon\{1,2\}$ and

$$\left|\phi_1^{\pm}\right\rangle = \exp\left[\mp i\varepsilon k(\hat{x}_1 + \frac{\hat{p}_1}{2m}t_1)t_1\right]\cdot\exp\left[i\hbar\frac{\varepsilon^2k^2}{12m}t_1^3\right]|\varphi_1(0)\rangle\,, \tag{6}$$

$$\left|\phi_2^{\pm}\right\rangle = \exp\left\{\mp i\varepsilon k[\hat{x}_2 + \frac{\hat{p}_2}{2m}(t_3+t_2)](t_3-t_2)\right\}\cdot\exp\left[i\hbar\frac{\varepsilon^2k^2}{12m}(t_3-t_2)^3\right]|\varphi_2(0)\rangle\,. \tag{7}$$

As it is evident, state (4) exhibits crossed correlations between the atomic translation, the atomic internal dynamics and the field variables. Since the aim of this paper is to analyze the entanglement between the internal variables of the two atoms as a function of the interaction time, we now trace on the atomic translational and field variables, and we obtain the following reduced density matrix

$$\begin{aligned}\rho = Tr_{f,s_1,s_2}(|\psi(t_3)\rangle\,\langle\psi(t_3)|) = P_1\,|g_1\rangle\,|g_2\rangle\,\langle g_1|\,\langle g_2| + \\ +P_2\{c_2^2\,|e_1\rangle\,|g_2\rangle\,\langle e_1|\,\langle g_2| + c_1^2\,|g_1\rangle\,|e_2\rangle\,\langle g_1|\,\langle e_2| + c_1c_2[q\,|e_1\rangle\,|g_2\rangle\,\langle g_1|\,\langle e_2| + h.c.]\}\end{aligned} \tag{8}$$

where $P_1 + P_2 = 1$ and

$$P_1 = \frac{1}{4}(1+c_R^{(1)})(1+c_R^{(2)}), \quad c_1 = \sqrt{\frac{(1+c_R^{(1)})(1-c_R^{(2)})}{4P_2}}, \quad c_2 = \sqrt{\frac{(1-c_R^{(1)})}{2P_2}}. \tag{9}$$

Apart from the complex parameter

$$q = i\frac{c_- - c_+}{\sqrt{2(1-c_R^{(1)})(1+c_R^{(1)})(1-c_R^{(2)})}}c_I^{(1)}, \tag{10}$$

equation(8) is formally very similar to the corresponding equation of Ref.[4]. However, our approach involves the entanglement with the atomic translation whose effects are encoded in the coefficients

$$c_R^{(i)} = Re(\langle\phi_i^+|\phi_i^-\rangle), \quad c_I^{(i)} = Im(\langle\phi_i^+|\phi_i^-\rangle), \quad c_\pm = \langle\phi_2^\pm|\varphi_2(0)\rangle\,. \tag{11}$$

The scalar products which appear in this equation play a crucial role in determining the separability of ρ. They are characterized, in fact, by a non dissipative damping term whose origins go back to the distance in the phase space of the translational components [14, 15, 16]. For instance, we found

$$\langle\phi_1^+|\phi_1^-\rangle = \exp(-\frac{d^2}{8})\exp(2ix_1\varepsilon kt_1), \tag{12}$$

where

$$d^2 = \frac{[x_1^+(t_1) - x_1^-(t_1)]^2}{\sigma_{x_1}^2} + \frac{[p_1^+(t_1) - p_1^-(t_1)]^2}{\sigma_{p_1}^2} \tag{13}$$

may be interpreted as the square of an adimensional distance in the phase space defined in terms of $x_1^{\pm}(t) = x_1 \mp a\, t^2/2$ and $p_1^{\pm}(t) = \mp m\, a\, t$ (with $a = \frac{\hbar k \epsilon}{m}$) and measured in units of σ_x and σ_p, along x and p, respectively. Similar behaviors hold for the other scalar products. In deriving this and similar expressions we have considered, for both the atoms, Gaussian initial packets of minimum uncertainty, centered in x_i with zero mean velocity along the x-cavity axis and of width σ_{x_i} ($\sigma_{p_i} = \frac{\hbar}{2\sigma_{x_i}}$). If the damping factors involved in the scalar products of eq. (11) are disregarded, we recover the reduced state of Ref.[4], obtained in the ambit of the JC model

$$\rho = \cos^4(\varepsilon_{JC}T)\,|g_1\rangle\,|g_2\rangle\,\langle g_1|\,\langle g_2| + \\ + \sin^2(\varepsilon_{JC}T)(|e_1\rangle\,|g_2\rangle + \cos(\varepsilon_{JC}T)\,|g_1\rangle\,|e_2\rangle)\cdot(\langle e_1|\,\langle g_2| + \cos(\varepsilon_{JC}T)\,\langle g_1|\,\langle e_2|) \tag{14}$$

where, for simplicity, we have fixed the relation $x_1\varepsilon k = x_2\varepsilon k \equiv \varepsilon_{JC}$ and same interaction times for the two atoms, $t_3 - t_2 = t_1 = T$, have been considered. Our main result, concerning the effect of the translational dynamics on the interatomic correlations, can be seen in a simple way, with a straightforward comparison between the equations (8) and (14). After a few periods of Rabi oscillations, the damping factors involved in the scalar products (11)(see Eq.(12)), determine a decoherence of the density matrix (8), i.e. the off-diagonal terms go to zero. As it is known, a system is separable if it is possible to set the state in the form $\rho = \sum_{r=1}^{\infty} p_r\, W_r^{(1)} \otimes W_r^{(2)}$, where $W_r^{(1)}$ and $W_r^{(2)}$ are states corresponding to the single subsystem and p_r are probabilities ($\sum_{r=1}^{\infty} p_r = 1$). The Peres-Horodecki test [11, 17] says that the separability is assured by the non-negativity of the eigenvalues of the partial transposed matrix of (8). We get for these eigenvalues

$$\lambda_1 = c_1^2 P_2, \qquad \lambda_2 = c_2^2 P_2, \qquad \lambda_{\pm} = \frac{P_1}{2}(1 \pm \sqrt{1 + (2|q|c_1c_2P_2 \backslash P_1)^2}) \tag{15}$$

which become all non-negative for $q \to 0$. In other words, from eqs.(9-12) and subsequent comments, it is evident that, in our case, the system becomes separable for interaction times sufficiently large. *Viceversa* a straightforward application of the the same test to the system described by (14) gives the condition

$$\sin^2(2\varepsilon_{JC}T)\sin^2(\varepsilon_{JC}T) = 0 \tag{16}$$

for the separability. Evidently, the periodical nature of the correlations in the JC model context implies an essential non separability of the system.

It is also possible to investigate the nature of the interatomic correlations in terms of the Bell's inequality. Because of its simplicity, we consider the Horodecki family formulation [18], which is equivalent to the standard Clauser, Horne, Shimony, Holt, (CHSH) formulation [19], when a bipartite system of spin 1/2 is involved, as in our case. The test reads: *A density matrix ρ describing a system composed by two spin 1/2 subsystems violates some Bell's inequality in the CHSH formulation if and only if the relation $M(\rho) > 1$ is satisfied*. The quantity $M(\rho)$ can be defined as follows. Consider the 3×3 matrix T_ρ with coefficients $t_{n,m} = tr(\rho\, \sigma_n \otimes \sigma_m)$, where σ_n are the standard Pauli matrices. Diagonalizing the symmetric matrix $U_\rho = T_\rho^T \cdot T_\rho$ (T_ρ^T is the transpose of T_ρ), and denoting the three eigenvalues of U_ρ by ν_1, ν_2 and ν_3, then $M(\rho) = max\{\nu_1 + \nu_2, \nu_1 + \nu_3, \nu_2 + \nu_3\}$. For the initial state (3) there is a degenerate eigenvalue, $\nu_2 = \nu_3$ and as a consequence $M(\rho) = max\{\nu_1 + \nu_2, 2\nu_2\}$. Fig. 1 compares the behaviors of $\nu_1 + \nu_2$ (continuous line) and $2\nu_2$ (dashed line) as a function of the interaction time for the two models. For simplicity, in both the figures (i) and (ii) we have assumed $t_3 - t_2 = t_1 = T$ and $t_2 - t_1 = T$. The response of the Bell's inequality test outlines the difference between the interatomic correlations predicted by the two models. When the translational dynamics is included in the quantum treatment of the system, a non dissipative damping of the quantum correlations appears, which can be ascribed to a which-way decoherence effect. These effects have actually been extensively analyzed in other context [16, 20, 21, 22, 23, 24, 25], both in theoretical and experimental works. It has here been shown how the complementarity aspect related to each atom may weaken the entanglement of an EPR state, leading towards classical correlations between the two atoms.

It is possible, furthermore, to extend the discussion to another simple case defined by the initial state $|\psi(0)\rangle = |e_1\rangle\,|g_2\rangle\,|0\rangle\,|\varphi_1(0)\rangle\,|\varphi_2(0)\rangle$. For this initial state, the quantity $M(\rho)$ reduces simply to the dashed line of both the figures (i) and (ii). As it is clear, the violation predicted by the JC model disappears for all the interaction times, when the translational dynamics is taken into account.

The behaviors shown in Fig. 1 suggest that there can be some difficulties in attaining a violation of Bell's inequality for massive particles, at least in the conditions here considered. It is to notice that at the heart of the damping of correlations there are the scalar products $\langle \phi_i^+ | \phi_i^- \rangle$ and $\langle \phi_i^{\pm} | \varphi_i(0) \rangle$, as it is clear from eq.s (8), (11) and (12). As it has been shown in Ref.[15], similar behaviors of the scalar products are found when the atoms enter the cavity near an antinodal point of the field mode. As a consequence it would be worth to test if our conceptual results can be extended to the antinodal region.

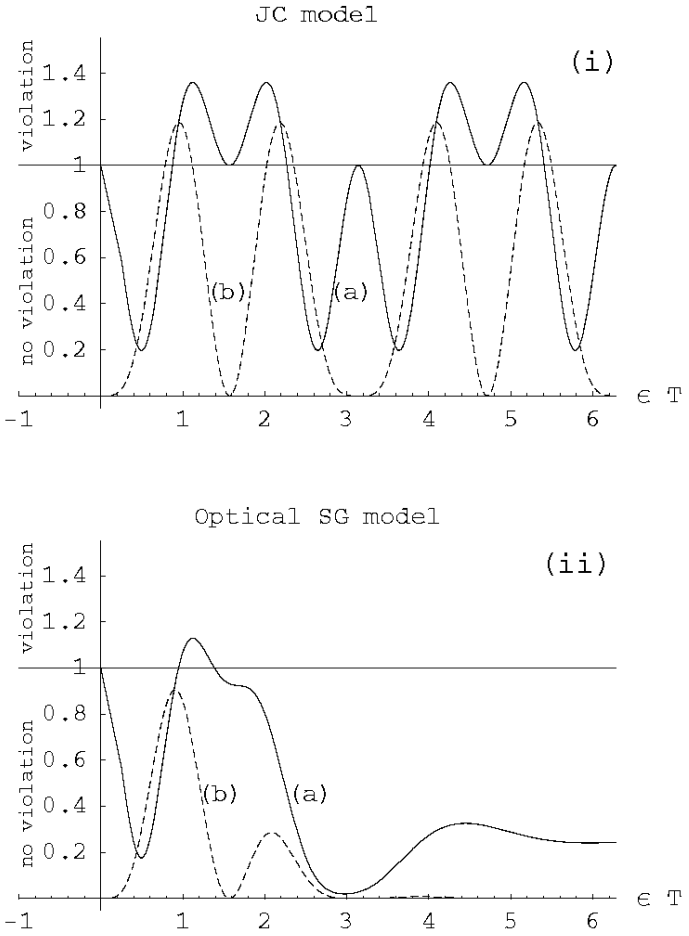

FIG. 1: Graphical solution of Bell's inequality in terms of the $M(\rho) = max\{(a), (b)\}$ function, for two two-level atoms interacting in succession with the field of the same cavity. Figure (i) shows the periodicity of $\nu_1 + \nu_2$ (continuous line)and $2\nu_2$ (dashed line) when the JC model is adopted and it reproduce the results of Ref.[4]. Fig. (ii) illustrates the non dissipative damping of the corresponding quantities due to the entanglement of the field and the internal atomic variables with the translational atomic degrees of freedom. Concerning the translational dynamics we suppose for both the atoms an initial wave packet of minimum uncertainty, with zero mean value of $\hat{p}_1$ and $\hat{p}_2$, centered in $x_1 = x_2 = \lambda/10$ and with a width $\sigma_{x_1} = \sigma_{x_2} = \lambda/10$, where $\lambda = 2\pi/k$ is the wavelength of the resonant k-mode of the undamped cavity. The values of the other parameters are $m = 10^{-26}$ kg and $\lambda = 10^{-5}$ meters. As the interaction time increases, the scalar products appearing in Eq.(11) become negligible and the nonlocality of the quantum correlations between the internal variables of the two atoms is seriously compromised.

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